Landscape of separable covariance matrices

Random matrices beyond sample covariance matrices

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Joint work with Fan Yang (Upenn)

Model from matrix normal distribution

- $Y = (y_{ij}), i = 1, 2, \cdots, p; j = 1, 2, \cdots n.$
- Matrix normal distribution (De Waal, 1985)

$$\mathbb{E}\,\mathrm{Vec}(Y) = \mathbf{0}_{pn},\ \mathrm{Cov}(\mathrm{Vec}(Y)) = A \otimes B.$$

- A : spatial; B : temporal.
- Spatial-temporal process: Climate, environmental sciences, medical sciences, brain-imaing.
- $Y = A^{1/2} X B^{1/2}, X \in \mathbb{R}^{p \times n}$ are of i.i.d. entries.
- $B = I \to$ sample covariance matrix .

Basic assumptions

- $X = (x_{ij}) \in \mathbb{R}^{p \times n}$ with i.i.d. entries and

$$\mathbb{E}x_{ij} = 0, \ \mathbb{E}x_{ij}^2 = n^{-1}.$$

- For some large integer $\varsigma > 0$ such that for $1 \leq k \leq \varsigma$

$$\mathbb{E}|x_{ij}|^k \leq C_k n^{-k/2}.$$

- For some constant $0 < \tau \leq 1$, we have

$$\tau \leq d := p/n \leq \tau^{-1}.$$

- $A \in \mathbb{R}^{p \times p}$ and $B \in \mathbb{R}^{n \times n}$ are some p.d.f. deterministic matrices satisfying regularity assumptions.

$$\operatorname{Spec}(A) = \{a_1, a_2, \cdots, a_p\}, \ \operatorname{Spec}(B) = \{b_1, b_2, \cdots, b_n\}.$$

Some notations

- $\mathcal{Q}_1 = A^{1/2} X B X^\top A^{1/2}, \mathcal{Q}_2 = B^{1/2} X^\top A X B^{1/2}$

$$\mathcal{G}_1(z) = (\mathcal{Q}_1 - z)^{-1},\ \mathcal{G}_2(z) = (\mathcal{Q}_2 - z)^{-1}.$$

- ρ : ESD of $\mathcal{Q}_1$. Stieltjes transform of ρ

$$m(z) = p^{-1} \operatorname{tr} \mathcal{G}_1(z).$$

$$m_1(z) = n^{-1} \sum_{i=1}^{p} a_i (\mathcal{G}_1(z))_{ii},\ m_2(z) = n^{-1} \sum_{i=1}^{n} b_i (\mathcal{G}_2(z))_{ii}(z).$$

- Self-consistent equations: $(m_{1c}, m_{2c}) \in \mathbb{C}_+^2$

$$m_{1c}(z) = d \int \frac{x}{-z(1 + x m_{2c}(z))} \pi_A(dx)$$

$$m_{2c}(z) = d \int \frac{x}{-z(1 + x m_{1c}(z))} \pi_B(dx)$$

Macroscopic picture

- Define $m_c(z)$

$$m_c(z) = d \int \frac{1}{-z(1 + x m_{2c}(z))} \pi_A(dx).$$

Theorem (Zhang, 2007)

For any $z \in \mathbb{C}_+$, there exists a unique solution $(m_{1c}, m_{2c}) \in \mathbb{C}_+^2$ to the systems of self-consistent equations. The function m_c is the Stieltjes transform of a probability measure μ_c supported on $\mathbb{R}_+$. Moreover, μ_c has a continuous derivative $\rho_c(x)$ on $(0, \infty)$.

- Edge behavior of ρ_c

$$f(z, m) := -m + \int \frac{x}{-z + xd \int \frac{t}{1+tm} \pi_A(dt)} \pi_B(dx).$$

Macroscopic picture: D. and Yang, 2019

- The densities ρ_c and $\rho_{1,2c}$ all have the same support on $(0, \infty)$, which is a union of intervals:

$$\operatorname{supp} \rho_c \cap (0, \infty) = \operatorname{supp} \rho_{1,2c} \cap (0, \infty) = \bigcup_{k=1}^{q} [\alpha_{2k}, \alpha_{2k-1}] \cap (0, \infty),$$

  where $q \in \mathbb{N}$ depends only on $\pi_{A,B}$.

- $(x, m) = (\alpha_k, m_{2c}(\alpha_k))$ are the real solutions to the equations

$$f(x, m) = 0, \quad \text{and} \quad \frac{\partial f}{\partial m}(x, m) = 0.$$

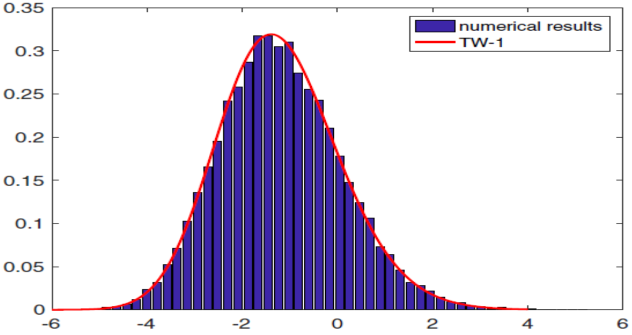
  Moreover, we have $m_{1c}(\alpha_1) \in (-b_1^{-1}, 0)$ and $m_{2c}(\alpha_1) \in (-a_1^{-1}, 0)$.

What do we expect?

- Setup: $d = 0.5$ and $n = 400$

$$\pi_A, \pi_B = 0.5\mathbf{1}_1 + 0.5\mathbf{1}_4$$

- Normalization: (El Karoui , AOP 2007)



Regularity assumption: D. and Yang, 2019

- $\lambda_r := \alpha_1$. To ensure TW, we expect: there exist constants $\beta_{1,2} > 0$ such that when $\operatorname{Im} z \geq 0$,

$$\rho_{1,2c}(\lambda_r - x) = \beta_{1,2} x^{1/2} + O(x), \quad x \downarrow 0,$$

$$m_{1,2c}(z) = m_{1,2c}(\lambda_r) + \pi a_{1,2}(z - \lambda_r)^{1/2} + O(|z - \lambda_r|), \; z \to \lambda_r.$$

- Taylor expansion: $\partial_z f(\lambda_r, m_r), \partial_\alpha^2 f(\lambda_r, m_r) = O(1)$.
- **Regularity assumption**

$$1 + m_{1c}(\lambda_r) b_1 \geq \tau, \; 1 + m_{2c}(\lambda_r) a_1 \geq \tau.$$

Separable sample covariance matrices without spikes

Theorem (D. and Yang, 2019+)

For the separable sample covariance matrices $\mathcal{Q}_1$, there exists some constant γ_r such that

$$n^{2/3}\gamma_r(\lambda_1(\mathcal{Q}_1)-\lambda_r) \Rightarrow \mathrm{TW}\,.$$

- Joint distributions: for some fixed constant $K>0$,

$$(n^{2/3}\beta_r(\lambda_1-\lambda_r),\cdots,n^{2/3}\beta_r(\lambda_K-\lambda_r)) \asymp (n^{2/3}(\lambda_1^{\mathrm{GOE}}-2),\cdots,n^{2/3}(\lambda_K^{\mathrm{GOE}}-2))$$

- The convergent limits and rates are first established in (Yang, EJP 2019). More generally, we have the rigidity results

$$|\lambda_i(\mathcal{Q}_1)-\gamma_i| \prec \min\{i, n\wedge p-i+1\}^{-1/3}n^{-2/3}.$$

To add spikes, we assume that there exist some fixed intergers $r, s \in \mathbb{N}$ and constants d_i^a, $1 \le i \le r$, and d_μ^b, $1 \le \mu \le s$, such that

$$\widetilde{A} = V^a \widetilde{\Sigma}^a (V^a)^\top, \ \widetilde{B} = V^b \widetilde{\Sigma}^b (V^b)^\top,$$

and

$$\widetilde{\Sigma}^a = \mathrm{diag}(\widetilde{a}_1, \ldots, \widetilde{a}_p), \ \widetilde{\Sigma}^b = \mathrm{diag}(\widetilde{b}_1, \ldots, \widetilde{b}_n).$$

Here

$$\widetilde{a}_i = \begin{cases} a_i(1 + d_i^a), & 1 \le i \le r \\ a_i, & \text{otherwise} \end{cases}, \qquad \widetilde{b}_\mu = \begin{cases} b_\mu(1 + d_\mu^b), & 1 \le \mu \le s \\ b_\mu & \text{otherwise} \end{cases}.$$

Spike separable sample covariance matrices: outliers

- We assume the followings hold for all $1 \leq i \leq r$ and $1 \leq \mu \leq s$.

$$\widetilde{a}_i > -m_{2c}^{-1}(\lambda_r) \quad \text{or} \quad \widetilde{b}_i > -m_{1c}^{-1}(\lambda_r).$$

- We define the integers $0 \leq r^+ \leq r$ and $0 \leq s^+ \leq s$ such that

$$\widetilde{a}_i \geq -m_{2c}^{-1}(\lambda_r) + n^{-1/3} \quad \text{if and only if} \quad 1 \leq i \leq r^+,$$

and

$$\widetilde{b}_\mu \geq -m_{1c}^{-1}(\lambda_r) + n^{-1/3} \quad \text{if and only if} \quad 1 \leq \mu \leq s^+.$$

Spike separable sample covariance matrices: outliers

- Deterministic locations

$$\theta_1(\widetilde{a}_i) := g_{2c}\left(-a_i^{-1}\right) \quad \text{or} \quad \theta_2(\widetilde{b}_\mu) := g_{1c}\left(-\widetilde{b}_\mu^{-1}\right),$$

  where g_{1c}, g_{2c} are respectively the inverse functions of $m_{1c} : (\lambda_r, \infty) \to (m_{1c}(\lambda_r), 0), m_{2c} : (\lambda_r, \infty) \to (m_{2c}(\lambda_r), 0)$.
- We define the labelling functions $\alpha : \{1, \cdots, p\} \to \mathbb{N}$ and $\beta : \{1, \cdots, n\} \to \mathbb{N}$ as follows. For any $1 \leq i \leq r$, we assign to it a label $\alpha(i) \in \{1, \cdots, r+s\}$ if $\theta_1(\widetilde{a}_i)$ is the $\alpha(i)$-th largest element in $\{\theta_1(\widetilde{a}_i)\}_{i=1}^r \cup \{\theta_2(\widetilde{b}_\mu)\}_{\mu=1}^s$. We also assign to any $1 \leq \mu \leq s$ a label $\beta(\mu) \in \{1, \cdots, r+s\}$ in a similar way. Moreover, we define $\alpha(i) = i + s$ if $i > r$ and $\beta(\mu) = \mu + r$ if $\mu > s$.

Spike separable sample covariance matrices: outliers

- Index notations

$$\mathcal{O} := \{\alpha(i) : 1 \le i \le r\} \cup \{\beta(\mu) : 1 \le \mu \le s\},$$

$$\mathcal{O}^+ := \{\alpha(i) : 1 \le i \le r^+\} \cup \{\beta(\mu) : 1 \le \mu \le s^+\}.$$

- Fluctuation level

$$\Delta_1(\widetilde{a}_i) := \left(\widetilde{a}_i + m_{2c}^{-1}(\lambda_r)\right)^{1/2}, \ \Delta_2(b_\mu) := \left(\widetilde{b}_\mu + m_{1c}^{-1}(\lambda_r)\right)^{1/2}.$$

Outlier and extremal non-outlier eigenvalues

Theorem (D. and Yang, 2019)

$$\left|\widetilde{\lambda}_{\alpha(i)} - \theta_1(\widetilde{a}_i)\right| \prec n^{-1/2}\Delta_1(\widetilde{a}_i), \quad 1 \le i \le r^+,$$

$$\left|\widetilde{\lambda}_{\beta(\mu)} - \theta_2(\widetilde{b}_\mu)\right| \prec n^{-1/2}\Delta_2(\widetilde{b}_\mu), \quad 1 \le \mu \le s^+.$$

Furthermore, for any fixed $\varpi > r + s$, we have

$$|\widetilde{\lambda}_i - \lambda_r| \prec n^{-2/3}, \quad \text{for } \; i \notin \mathcal{O}^+ \text{ and } \; i \le \varpi.$$

- When Δ_1 changes from $n^{-1/3}$ to $O(1)$, we expect a transition for TW to Gaussian.
- (Bao-D.-Wang, 2018), (Bao-D.-Wang-Wang, 2019).
- In general, variance depends on 4th cumulants (both third and fourth moments).

Eigenvalue sticking

Theorem (D. and Yang, 2019)

$$\alpha_+ := \min\left\{ \min_i \left|\widetilde{a}_i + m_{2c}^{-1}(\lambda_r)\right|, \min_\mu \left|\widetilde{b}_\mu + m_{1c}^{-1}(\lambda_r)\right| \right\}.$$

Fix any sufficiently small constant $\tau > 0$. We have that for $1 \le i \le \tau n$,

$$\left|\widetilde{\lambda}_{i+r+s} - \lambda_i\right| \prec \frac{1}{n\alpha_+}, \quad 1 \le i \le \tau n.$$

- Tracy-Widom.

Outlier singular vectors

For $1 \leq i \leq r^+$, $1 \leq j \leq p$ and $1 \leq \nu \leq n$, we define

$$\delta^a_{\alpha(i),\alpha(j)} := |\widetilde{a}_j - \widetilde{a}_i|, \quad \delta^a_{\alpha(i),\beta(\nu)} := \left|\widetilde{b}_\nu + m_{1c}^{-1}(\theta_1(\widetilde{a}_i)\right|.$$

Similarly, for $1 \leq \mu \leq s^+$, $1 \leq j \leq p$ and $1 \leq \nu \leq n$, we define

$$\delta^b_{\beta(\mu),\alpha(j)} := |\widetilde{a}_j + m_{2c}^{-1}(\theta_2(\widetilde{b}_\mu))|, \quad \delta^b_{\beta(\mu),\beta(\nu)} := |\widetilde{b}_\nu - \widetilde{b}_\mu|.$$

Denote
$\delta_{\alpha(i)} = \left(\min_{k:\alpha(k)\neq\alpha(i)} \delta^a_{\alpha(i),\alpha(k)}\right) \wedge \left(\min_{\mu:\beta(\mu)\neq\alpha(i)} \delta^a_{\alpha(i),\beta(\mu)}\right)$

Outlier singular vectors

Theorem (D. and Yang, 2019)

$$|\langle \mathbf{v}_i^a, \widetilde{\boldsymbol{\xi}}_i\rangle|^2 =$$

$$\frac{1}{\widetilde{a}_i}\frac{g_{2c}'(-(\widetilde{a}_i)^{-1})}{g_{2c}(-(\widetilde{a}_i)^{-1})} + O_\prec\left(\frac{1}{n^{1/2}(\widetilde{a}_i + (m_{2c}^{-1}(\lambda_r))^{1/2}} + \frac{1}{n\delta_{\alpha(i)}^2}\right).$$

- Non-overlapping condition

$$\widetilde{a}_i + m_{2c}^{-1}(\lambda_r) \gg n^{-1/3}, \quad \delta_{\alpha(i)} \gg (\widetilde{b}_i + m_{2c}^{-1}(\lambda_r))^{-1/2} n^{-1/2}.$$

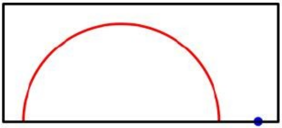
Theorem (D. and Yang, 2019)

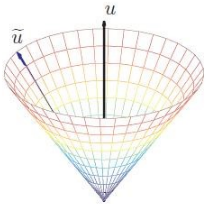
If $\alpha(i) \notin \mathcal{O}^+$, we have

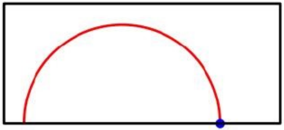
$$|\langle \mathbf{v}_j^a, \widetilde{\boldsymbol{\xi}}_{\alpha(i)}\rangle|^2 \prec \frac{1}{n(|\widetilde{a}_j + m_{2c}^{-1}(\lambda_r)|^2 + n^{-2/3})}.$$

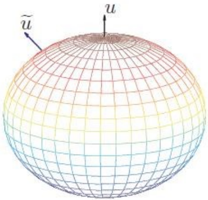
- Right singular vectors. General components.

General picture



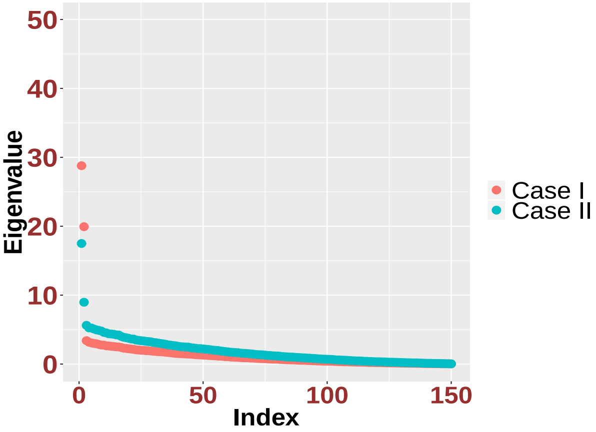




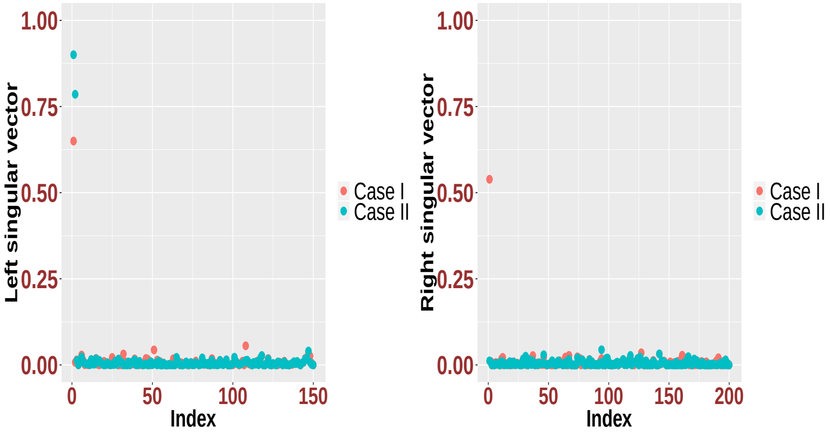


Some statistical remarks: which model?

$$\widetilde{\Sigma}^a = \text{diag}(5, 1, \cdots, 1), \quad \widetilde{\Sigma}^b = \text{diag}(5, 1, \cdots, 1), \qquad \text{(Case I)}$$
$$\widetilde{\Sigma}^a = \text{diag}(3, 2, 1, \cdots, 1), \quad \widetilde{\Sigma}^b = \text{diag}(1, 1, \cdots, 1). \qquad \text{(Case II)}$$



Some statistical remarks: which model?



Some statistical remarks: adaptive estimation

$$\widehat{a}_i := -\left(\frac{1}{n}\sum_{\nu=r+s+1}^{n}\frac{1}{\widetilde{\lambda}_\nu(\widetilde{\mathcal{Q}}_2)-\widetilde{\lambda}_{\alpha(i)}}\right)^{-1}, \quad 1 \leq i \leq r+s.$$

$$\widehat{b}_\mu := -\left(\frac{1}{n}\sum_{k=r+s+1}^{p}\frac{1}{\widetilde{\lambda}_k(\widetilde{\mathcal{Q}}_1)-\widetilde{\lambda}_{\beta(\mu)}}\right)^{-1}, \quad 1 \leq \mu \leq r+s.$$

- Suppose $\widetilde{B} = I_n + \mathcal{M}_n$, where $\mathcal{M}_n$ is a matrix of rank l_n. Then we have that for $1 \leq i \leq r$,

$$\widetilde{a}_i = \widehat{a}_i + O_\prec(l_n n^{-1/2}).$$

  Similarly, if $\widetilde{A}$ is an l_n-rank perturbation of the identity matrix, then for $1 \leq \mu \leq s$,

$$\widetilde{b}_\mu = \widehat{b}_\mu + O_\prec(l_n n^{-1/2}).$$

Anisotropic local laws: (Yang, 2019) and (D. and Yang, 2019)

- $(p+n)\times(p+n)$ self-adjoint block matrix (linear function of X):

$$H \equiv H(X,z) := z^{1/2}\begin{pmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{pmatrix}, \quad z \in \mathbb{C}_+.$$

$$G \equiv G(X,z) := \left(H(X,z) - z\right)^{-1}.$$

- Schur complement formula

$$G(z) = \begin{pmatrix} \mathcal{G}_1 & z^{-1/2}\mathcal{G}_1 Y \\ z^{-1/2}Y^*\mathcal{G}_1 & \mathcal{G}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{G}_1 & z^{-1/2}Y\mathcal{G}_2 \\ z^{-1/2}\mathcal{G}_2 Y^* & \mathcal{G}_2 \end{pmatrix}$$

where $Y := A^{1/2}XB^{1/2}$.

Anisotropic local laws: (Yang, 2019) and (D. and Yang, 2019)

- Spectral domains

$$S(\varsigma_1, \varsigma_2) := \{z = E + \mathrm{i}\eta : \lambda_r - \varsigma_1 \le E \le \varsigma_2 \lambda_r,\ 0 < \eta \le 1\},$$

$$S_0(\varsigma_1, \varsigma_2, \epsilon) := S(\varsigma_1, \varsigma_2) \cap \left\{z = E + \mathrm{i}\eta : \eta \ge n^{-1+\epsilon}\right\},$$

$$z \in S_{out}(\varsigma_2, \epsilon) := \left\{E + \mathrm{i}\eta : \lambda_r + n^{-2/3+\varepsilon} \le E \le \varsigma_2 \lambda_r, \eta \in [0, 1]\right\}.$$

- Convergent limits and control parameters

$$\Pi(z) := \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix}, \ \Psi(z) := \sqrt{\frac{\operatorname{Im} m_{2c}(z)}{n\eta}} + \frac{1}{n\eta}.$$

$$\Pi_1 := -z^{-1}\left(1 + m_{2c}(z) A\right)^{-1}, \quad \Pi_2 := -z^{-1}(1 + m_{1c}(z) B)^{-1}.$$

Anisotropic local laws: (Yang, 2019) and (D. and Yang, 2019)

- For $z \in S_0(\varsigma_1, \varsigma_2, \epsilon)$, we have

$$|\langle \mathbf{u}, G(X,z)\mathbf{v}\rangle - \langle \mathbf{u}, \Pi(z)\mathbf{v}\rangle| \prec \Psi(z).$$

$$|m(z)-m_c(z)|+|m_1(z)-m_{1c}(z)|+|m_2(z)-m_{2c}(z)| \prec (n\eta)^{-1}.$$

- For $z \in S_{out}(\varsigma_2, \epsilon), \kappa = |\operatorname{Re} z - \lambda_r|$

$$|\langle \mathbf{u}, G(X,z)\mathbf{v}\rangle - \langle \mathbf{u}, \Pi(z)\mathbf{v}\rangle| \prec \sqrt{\frac{\operatorname{Im} m_{2c}(z)}{n\eta}} \asymp n^{-1/2}(\kappa+\eta)^{-1/4},$$

$$\begin{aligned} &|m(z)-m_c(z)| + |m_1(z)-m_{1c}(z)| + |m_2(z)-m_{2c}(z)| \\ &\prec \frac{1}{n(\kappa+\eta)} + \frac{1}{(n\eta)^2\sqrt{\kappa+\eta}}. \end{aligned}$$

Technical proofs—Universality

- Three ways to to prove universality (Erdos-Yau)
  1. Comparison method (Erdos-Yau-Yin, Adv, 2010), (Bao-Pan-Zhou, AOS 2015; D.-Yang, 2018, AOAP)
  2. Continuous interpolation (Schelli-Lee, AOAP, 2016) (Johnstone-Zhou, 2018, "discrete version", swapping pair)
  3. Dyson Brownian motion (Erodos-Yau, JAMS;Landu-Yau, 2018)
- Three-step-strategy using DM
  1. Local laws for the random matrix ensemble H+rigidity of eigenvalues: **Proved**
  2. University of $H_t = H + \sqrt{t}G$, G is GOE, $t = o(1)$. t is the time such that **local** eigenvalue statistics reach equilibrium. Needs some basic discussion from free probability theory.
  3. A density argument comparing the eigenvalue statistics between Step 2 and the random matrix ensemble.

Technical proofs–Universality

$$H_t := \begin{pmatrix} 0 & W+\sqrt{t}X \\ (W+\sqrt{t}X)^\top & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} D & 0 \end{pmatrix}, D^2 = \mathrm{diag}(d_1, \cdots, d_p).$$

- Denote μ_i as the unique strong solution to the SDE

$$d\mu_i = 2\sqrt{\mu_i}\frac{dB_i}{\sqrt{n}} + \left(\frac{1}{N} \sum_{l \neq k} \frac{\lambda_k + \lambda_l}{\lambda_k - \lambda_l} + \frac{p}{n} \right) dt,$$

  $\mu_i(0)$: Wishart matrices.
- For some i_0, we denote λ_i as

$$d\mu_i = 2\sqrt{\mu_i}\frac{dB_{i-i_0+1}}{\sqrt{n}} + \left(\frac{1}{N} \sum_{l \neq k} \frac{\lambda_k + \lambda_l}{\lambda_k - \lambda_l} + \frac{p}{n} \right) dt,$$

  with $\lambda_i(0) = \lambda_i(\gamma_0 H_{t_0})$

Techincal proofs-Universality

Theorem (D. and Yang, 2019+)

For $t_0 = n^{-1/3+\epsilon_0}$ *and* $t_1 = n^{-1/3+\epsilon_1}, 0 < \epsilon_1 < \epsilon_0/200,$

$$|(\lambda_{i_0+i-1}(t_1) - E_\lambda(t_1)) - (\mu_i - E_\mu(t_1))| \prec n^{-2/3-\delta},$$

for any bounded i.

- $E_\lambda(\cdot)$ is the edge. Clearly, $E_\mu(t) = (1+\sqrt{d})^2\sqrt{1+t}$.
- To find γ and $E_\mu()$, we need to study the macroscopic structure of signal-plus-noise matrix.
- Free probability: rectangular free convolution and subordiation function $\Rightarrow$ square root behavior.

Technical proofs–Outliers

$$\widetilde{H}(X,z) = z^{1/2}\begin{pmatrix} 0 & \widetilde{A}^{1/2}X\widetilde{B}^{1/2} \\ \widetilde{B}^{1/2}X^*\widetilde{A}^{1/2} & 0 \end{pmatrix}, \quad z \in \mathbb{C}_+ \cup \mathbb{R}.$$

$$\mathbf{U} = \begin{pmatrix} V_o^a & 0 \\ 0 & V_o^b \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} D^a(D^a+1)^{-1} & 0 \\ 0 & D^b(D^b+1)^{-1} \end{pmatrix}.$$

- Master equation

$$\det\left(\mathcal{D}^{-1} + x\mathbf{U}^* G(x)\mathbf{U}\right) = 0.$$

- Anisotropic local laws $\Rightarrow$

$$\det(\mathcal{D}^{-1} + x\mathbf{U}^*\Pi(x)\mathbf{U}) = 0 \quad \Rightarrow$$

$$\prod_{i=1}^{r}\left(\frac{d_i^a+1}{d_i^a} - \frac{1}{1+m_{2c}(x)\sigma_i^a}\right)\prod_{\mu=1}^{s}\left(\frac{d_\mu^b+1}{d_\mu^b} - \frac{1}{1+m_{1c}(x)\sigma_\mu^b}\right) = 0.$$

Outlier eigenvectors

$$\langle \mathbf{v}_i^a, \widetilde{\boldsymbol{\xi}}_i \mathbf{v}_j^a \rangle = -\frac{1}{2\pi \mathrm{i}} \oint_{g_{2c}(\Gamma)} \langle \mathbf{v}_i^a, \widetilde{G}(z) \mathbf{v}_j^a \rangle dz,$$

$$\mathbf{U}^* \widetilde{G}(z) \mathbf{U} = \widetilde{\mathcal{D}}^{1/2} \left[\mathbf{U}^* G(z) \mathbf{U} - z \mathbf{U}^* G(z) \mathbf{U} \frac{1}{\mathcal{D}^{-1} + z \mathbf{U}^* G(z) \mathbf{U}} \mathbf{U}^* G(z) \mathbf{U} \right] \widetilde{\mathcal{D}}^{1/2},$$

where

$$\widetilde{\mathcal{D}} := \begin{pmatrix} (1 + D^a)^{-1} & 0 \\ 0 & (1 + D^b)^{-1} \end{pmatrix}.$$

References

Xiucai Ding and Fan Yang (2019)
Spiked seperable sample covariance matrices and principal components

Xiucai Ding and Fan Yang (2019+)
Tracy-Widom distribution of seperable sample covariance matrices